‡6 Ptolemy's Backwardness

Further Evidence That Ptolemy Didn't Deduce His Parameters from Observations

by Hugh Thurston[1]

A The Backwards Approach

A1 Imagine that you are a physics student finding the specific heat of lead. You desperately want high marks for your experiment. You can easily look up the specific heat and calculate what the thermometer in your experiment should read. If you record this calculated temperature instead of actually reading the thermometer, your result will be Excellent. That would be cheating, of course; but students have been known to do it. This type of fraud is called "working backwards from the answer."

A2 Delambre 1819 (pp.lxvij-lxix) showed that Ptolemy did precisely this. In the *Syntaxis* (3.1), Ptolemy claimed that he calculated the length of the year from firsthand-observed equinox and solstice times and dates. These four "observations", long known to be highly inaccurate (most are over 30 hours late! [see below: ‡7 §C3 & fn 14]) and long suspected to be fabricated, were in fact obtained by working backwards from the answer: plugging a previously-known (Hipparchus's) value for the length of the year into earlier equinox & solstice times yields precisely the times and dates that Ptolemy said he observed. Robert R. Newton rediscovered this and found several other examples of backwards working by Ptolemy. (See R.Newton 1977.)[2]

B Successive Approximation

B1 In Books 10 and 11 of the *Syntaxis*, Ptolemy used a well-known technique called "successive approximation". We use this technique when we have a problem that we can't solve exactly. If we can, by one means or another, find an approximate solution, we use this approximation to find a closer approximation, then use this to find an even closer one, and so on.

[1] Hugh Thurston holds a PhD in mathematics from Cambridge University, England. He is currently professor emeritus of mathematics at the University of British Columbia, Vancouver, Canada. His original mathematical discoveries in ancient astronomy have appeared in *Griffith Observer*, *Archive for History of Exact Sciences*, & *Journal for the History of Astronomy*. Favorably reviewed by the Royal Society's Desmond King-Hele in *Nature 370*:339-340 (1994/8/4), Thurston's successful 1994 book, *Early Astronomy* (published thanks heavily to the interest and discrimination of Thos. von Foerster, Springer-Verlag's Math-Physics Senior Editor), is the broadest careful investigation of pre-modern astronomy carried out by a professional mathematician in over a century (since Delambre's early 19th century work), covering not only Greek & Babylonian astronomy but: Chinese, Indian, Arabic, Mayan, & Renaissance. Among others.

[2] Note added by DR: See also Rawlins 1987 p.236 item 5 for proof (by a quite independent approach) that Ptolemy's mean motion of Mercury existed before the data he adduces to prove it. (For van der Waerden's comment on this simple demonstration, see *DIO 1.1* ‡6 fn 37. See, too, *DIO 1.2* fnn 16&166, and *DIO 2.1* ‡3 §C15.)

B2 A wonderful successive-approximation scheme for finding square roots appears to go back to Archimedes. (See Heath 1921 2:325.) The method is nicely explained in the final appendix to R.Newton's final book (R.Newton 1985 p.255):
"Suppose we want the square root of a number N. Let n be any approximation to $\sqrt{N}$. Then $[n + (N/n)]/2$ is a better approximation. That is, we divide any approximation n into N and take the average of n and the quotient. The average is a better approximation. In fact, if n and N/n agree to k significant figures, the average is accurate to $2k$ significant figures. Thus the process converges rapidly."

B3 (The widely-believed convergence rule here stated is over-simplified. The actual rule is: if the error in n is e, then the error[3] in $[n + (N/n)]/2$ is $e^2/2n$.)

B4 If we use this method to compute the square root of 2, starting with 1 1/2 for the first approximation, we will get the successive approximations:

3/2, 17/12, 577/408

(In just 3 iterations, we have a value good to 1.5 parts in a million. Which explains why — though "successive approximation" sounds fuzzy — the technique is so powerfully attractive.)

B5 Note that the approximations get less and less round as the sequence continues. This phenomenon is characteristic of successive approximation.

C Ptolemy's Orbital Successive Approximations

C1 Ptolemy used successive approximation to find the eccentricities of the outer planets. He started with purported observations of three oppositions to the mean Sun. (You'll find illustrations, if you want them, in Thurston 1994E pp.166-167, Figs.6.30-6.32.) At such an opposition the planet, the centre C of its epicycle, and the Earth T are in one straight line. (See *ibid* Fig.6.30.) Therefore, if C_1, C_2, & C_3 are the three opposition-positions of C, and E is the equant-point, then (since motion is uniform around E in the equant model Ptolemy adopted), the time-intervals between the oppositions give the angles C_1EC_2 and C_2EC_3. (See *ibid* Fig.6.31.) The observed longitudes of the planet give the angles C_1TC_2 and C_2TC_3. (See *ibid* Fig.6.32.)

C2 Problem: to calculate the distance ET. Of course, we can't find absolute distances, only ratios. But Ptolemy took the radius of the circle on which C moves to be 60, enabling him to give a value to ET. The ratio of ET to this radius is sometimes called the eccentricity.[4]

C3 Solution: let Z_1, Z_2, & Z_3 be the points where the lines EC_1, EC_2, & EC_3 intersect another circle of radius 60 (the dashed circle in Fig.6.32 of *ibid*), whose center is E (the equant point). If Ptolemy knew the angles Z_1TZ_2, Z_2TZ_3, Z_1EZ_2, & Z_2EZ_3, then he could, by a long but straightforward piece of Euclid-plus-chord-table geometry, find ET. (See *Syntaxis* 10.7 or Thurston 1994E App.5 for details. See also Hill 1900 and Rawlins 1987 n.25.) He did know Z_1EZ_2 & Z_2EZ_3: they are equal to C_1EC_2, & C_2EC_3. But he didn't know Z_1TZ_2 & Z_2TZ_3. However, these two angles are not much different from C_1TC_2 & C_2TC_3, which for each outer planet are allegedly (§C1) known from opposition observations — e.g., for Mars (*Syntaxis* 10.7): $67°50'$ & $93°44'$.

C4 So he calculated what ET would be if Z_1TZ_2 & Z_2TZ_3 were $67°50'$ & $93°44'$, and he got ET = 13;7. (This is notational shorthand for 13 + 7/60, which, for a circle of radius 60, constitutes an eccentricity = [13 + 7/60]/60 = 0.219.) At the same time, he calculated the direction of the apogee. These two basic parameters aren't exact because his input data aren't exact. But they are close, because the data are close.

[3] Note that this is not an upper limit on the error but rather an exact expression for it.

[4] This eccentricity should not be confused with the eccentricity of an ellipse. If a Greek orbit is the best approximation to a Keplerian ellipse's longitudinal motion, then the Greek eccentricity will be twice the eccentricity of the elliptical orbit. E.g., for Mars: Greek eccentricity = 1/5; elliptic eccentricity = 1/10.

C5 Now knowing the basic parameters of this 1^{st}-approximation orbit, he knew the motion of C (on it) completely and could calculate anything he liked in this orbit, including Z_1TZ_2 & Z_2TZ_3. The results won't be exact, because the parameters aren't exact; but they'll be better than his first crude approximation of (§C3) setting them equal to C_1TC_2 & C_2TC_3.

C6 With these better values for Z_1TZ_2 & Z_2TZ_3, he repeated the §C3 calculation and now got a better value for eccentricity ET. This in turn led him to better values for Z_1TZ_2 & Z_2TZ_3, and these gave him a yet better value for ET. Here he stopped.

C7 He used the same method for Jupiter and Saturn, except that (because their orbits' eccentricities are much lower than Mars') he needed to compute only two approximation-iterations instead of three.

D Ptolemy's Roundings: in Reverse

D1 As the steps progress, do Ptolemy's approximations get less round, as they should? They do not. The values for ET are (*Syntaxis* 10.7, 11.1, 11.5, respectively):

Mars 13 7/60, 11 5/6, 12. Jupiter 5 23/60, 5 1/2. Saturn 7 2/15, 6 5/6.

D2 No-one who has done much successive approximation will find these results plausible. They are what would be expected in working backwards from the neatly rounded answers.

References

J. B. J. Delambre 1819. *Histoire de l'Astronomie du Moyen Age*, Paris.
Thos.Heath 1921. *History of Greek Mathematics*, Oxford U.
Geo.Hill 1900. AJ *21*:33.
Karl Manitius 1912-1913, Ed. *Claudius Ptolemäus Handbuch der Astronomie*, Leipzig.
R.Newton 1977. *Crime of Claudius Ptolemy*, Johns Hopkins U.
R.Newton 1985. *Origins of Ptolemy's Astronomical Tables*, U.Maryland.
D.Rawlins 1987. American Journal of Physics *55*:235. [Note *DIO 11.2* §G & fnn 26-27.]
Syntaxis. Compiled by C.Ptolemy c.150 AD. Eds: Manitius 1912-3; Toomer 1984.
Hugh Thurston 1994E. *Early Astronomy*, NYC.
Gerald Toomer 1984, Ed. *Ptolemy's Almagest*, NYC.
B. L. van der Waerden 1963. *Science Awakening I* (Tr. Arnold Dresden), NYC.